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# Feigenbaum theory for unimodal maps with asymmetric critical point

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**Abstract.** We extend the renormalization analysis for period doubling in unimodal maps to the case of asymmetric critical points. Universal scaling phenomena are governed by period-two points of a renormalization operator.

## 1. Introduction

We are interested in exploring the universality classes of the period-doubling phenomenon as first expounded by Feigenbaum [9, 10] (see also [5]). The universality of the remarkable scaling phenomena observed in unimodal maps of the interval are explained in terms of a fixed point of the operator $f(x) \mapsto \alpha^{-1} f(f(\alpha x))$, where $\alpha = f(1)$. Following Feigenbaum's initial discoveries, a computer-assisted proof of the existence of the fixed point was given by Lanford [13], and since then various proofs of both existence and properties of the fixed point have been given [3, 4, 8, 7].

There has been much numerical and analytic work on how the scaling phenomena depend on the degree of the critical point. For a review of results see [16] and references therein. In particular the (computer-assisted) rigorous results of Eckmann and Wittwer [7] for the large-degree limit are noteworthy. The degree is seen to be a universality class parameter, by which we mean that different degrees correspond to different universal scalings.

In this paper we study unimodal maps of the interval possessing degree $d$ critical points, but with differing left- and right-hand limits of the $d$th derivative. The prototype map we have in mind is

$$f(x) = \begin{cases} f_L(x) = 1 - \lambda_1 |x|^d & x \leqslant 0 \\ f_R(x) = 1 - \lambda_2 |x|^d & x \geqslant 0 \end{cases} \tag{1.1}$$

with $d > 1$. The subscripts $L$ and $R$ stand for 'left' and 'right', and $\lambda_1, \lambda_2 > 0$. Early studies of period doubling in maps of this form were conducted by Arneodo *et al* [1] (see also [6, 15]). As observed in [1], the ratio $\mu$, of the coefficients $\lambda_1$ and $\lambda_2$ is another universality class parameter.

Feigenbaum's renormalization operator is

$$T_F : f \mapsto \tilde{f} \tag{1.2}$$

defined by

$$\tilde{f}(x) = \alpha^{-1} f f(\alpha x) \tag{1.3}$$

with

$$\alpha = f(1) < 0. \tag{1.4}$$

The natural generalization of this operator to the type of map we consider is the map on pairs of functions.

$$T : (f_L, f_R) \mapsto (\tilde{f}_L, \tilde{f}_R) \tag{1.5}$$

defined by

$$\tilde{f}_L(x) = \alpha^{-1} f_R f_R(\alpha x) \tag{1.6a}$$

$$\tilde{f}_R(x) = \alpha^{-1} f_R f_L(\alpha x) \tag{1.6b}$$

with

$$\alpha = f_R(1) < 0. \tag{1.7}$$

The analysis of section 2 below shows that it is natural to look for fixed points of the *second* iterate of $T$. We see in section 3 that indeed the parameter $\mu$ governs the universal scaling features. We expect there to be a line of period-two points parametrized by $\mu$ through the Feigenbaum fixed point, each point on the line corresponding to a universality class with its own universal constants. In section 4 we present numerical calculations of the relevant scaling exponents which support this conclusion.

## 2. Mixed degree maps

It is illustrative to consider not just the situation in which the maps join up with different left and right $d$th derivatives at the critical point as in equation (1.1), but one where the degree of the critical point may be different on the left- and right-hand sides. The prototype would now be

$$f(x) = \begin{cases} f_L(x) = 1 - \lambda_1 |x|^{d_1} & x \leqslant 0 \\ f_R(x) = 1 - \lambda_2 |x|^{d_2} & x \geqslant 0 \end{cases} \tag{2.1}$$

with $d_1, d_2 > 1$. For simplicity of the expressions we have assumed in what follows that the degrees are even, but this is not essential.

We consider the formal effect of the renormalization operator $T$ defined by (1.5) on the more general map with

$$f_L(x) = 1 + \sum_{i=1}^{\infty} a_i x^{i d_1} \tag{2.2a}$$

$$f_R(x) = 1 + \sum_{i=1}^{\infty} b_i x^{i d_2} \tag{2.2b}$$

in which $a_1, b_1 < 0$. Our aim is to find relations between the coefficients and scaling factors that are necessary for $T$ to have fixed and period-two points.

We have

$$\tilde{f}_L(x) = \alpha^{-1}\left(1 + \sum_{i=1}^{\infty} b_i \left(1 + \sum_{j=1}^{\infty} b_j \alpha^{j d_2} x^{j d_2}\right)^{i d_2}\right) \tag{2.3a}$$

$$\tilde{f}_R(x) = \alpha^{-1}\left(1 + \sum_{i=1}^{\infty} b_i \left(1 + \sum_{j=1}^{\infty} a_j \alpha^{j d_1} x^{j d_1}\right)^{i d_2}\right). \tag{2.3b}$$

We now seek to write these in the form

$$\tilde{f}_L(x) = 1 + \sum_{\ell=1}^{\infty} \tilde{a}_\ell x^{\ell d_2} \tag{2.4a}$$

$$\tilde{f}_R(x) = 1 + \sum_{\ell=1}^{\infty} \tilde{b}_\ell x^{\ell d_1}. \tag{2.4b}$$

Note that the degrees swop from the left- to right-hand side and vice versa. This leads us to expect that the second iterate $T^2$ of the renormalization operator will govern any scaling behaviour.

Let us consider $\tilde{f}_R$ first. We have

$$\tilde{f}_R(x) = \alpha^{-1} + \alpha^{-1} \sum_{i=1}^{\infty} b_i \sum_{k=0}^{id_2} \binom{id_2}{k} \left( \sum_{j=1}^{\infty} a_j \alpha^{jd_1} x^{jd_1} \right)^k \tag{2.5}$$

$$= \alpha^{-1} + \alpha^{-1} \sum_{k=0}^{\infty} \sideset{}{'}\sum_{i=\left[\frac{k+d_2-1}{d_2}\right]}^{\infty} b_i \binom{id_2}{k} \left( \sum_{j=1}^{\infty} a_j \alpha^{jd_1} x^{jd_1} \right)^k \tag{2.6}$$

on swopping the first two summations, where $\Sigma'$ means that when $k = 0$ there is no $i = 0$ term. This is

$$\tilde{f}_R(x) = \alpha^{-1} + \alpha^{-1} \sum_{k=0}^{\infty} \left( \sum_{j=1}^{\infty} a_j \alpha^{jd_1} x^{jd_1} \right)^k \sideset{}{'}\sum_{i=\left[\frac{k+d_2-1}{d_2}\right]}^{\infty} b_i \binom{id_2}{k} \tag{2.7}$$

$$= \alpha^{-1} + \alpha^{-1}(f_R(1) - 1) + \alpha^{-1} \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{j=1}^{\infty} a_j \alpha^{jd_1} x^{jd_1} \right)^k f_R^{(k)}(1) \tag{2.8}$$

where we have used the fact that the $k$th derivative of $f_R$, $f_R^{(k)}$, satisfies

$$\frac{1}{k!} f_R^{(k)}(1) = \sum_{i=\left[\frac{k+d_2-1}{d_2}\right]}^{\infty} \binom{id_2}{k} b_i. \tag{2.9}$$

Continuing (recalling that $\alpha = f_R(1)$), we thus have

$$\tilde{f}_R(x) = 1 + \alpha^{-1} \sum_{k=1}^{\infty} \frac{1}{k!} f_R^{(k)}(1) \left( \sum_{j=1}^{\infty} a_j \alpha^{jd_1} x^{jd_1} \right)^k \tag{2.10}$$

$$= 1 + \alpha^{-1} \sum_{k=1}^{\infty} \frac{1}{k!} f_R^{(k)}(1) \sum_{\ell=k}^{\infty} \left( \sum_{|i|=\ell} a_{i_1} a_{i_2} \ldots a_{i_k} \right) \alpha^{\ell d_1} x^{\ell d_1} \tag{2.11}$$

where the summation $\sum_{|i|=\ell} a_{i_1} a_{i_2} \ldots a_{i_k}$ means the sum of the product of $k$ terms with indices adding to $\ell$. We use the notation $|i| = i_1 + i_2 + \cdots + i_k$.

We now swop order of summation again to obtain

$$\tilde{f}_R(x) = 1 + \alpha^{-1} \sum_{\ell=1}^{\infty} \alpha^{\ell d_1} x^{\ell d_1} \sum_{k=1}^{\ell} \frac{1}{k!} f_R^{(k)}(1) \left( \sum_{|i|=\ell} a_{i_1} a_{i_2} \ldots a_{i_k} \right) \tag{2.12}$$

which is our desired expression (2.4*b*). For instance, the coefficient of $x^{d_1}$ is

$$\tilde{b}_1 = \alpha^{-1} \alpha^{d_1} \sum_{k=1}^{1} \frac{1}{k!} f_R^{(k)}(1) \left( \sum_{|i|=1} a_{i_1} a_{i_2} \ldots a_{i_k} \right) \tag{2.13}$$

$$= \alpha^{d_1 - 1} f_R'(1) a_1 \tag{2.14}$$

whilst

$$\tilde{b}_2 = \alpha^{2d_1-1}(f_R'(1)a_2 + \tfrac{1}{2}f_R''(1)a_1^2). \tag{2.15}$$

Note that the expression for each $\tilde{b}_\ell$ involves every $b_i$ (but only a finite number of the $a_i$). In general we have

$$\tilde{b}_\ell = \alpha^{\ell d_1 - 1}\sum_{k=1}^{\ell}\frac{1}{k!}f_R^{(k)}(1)\Bigg(\sum_{|i|=\ell}a_{i_1}a_{i_2}\ldots a_{i_k}\Bigg). \tag{2.16}$$

The corresponding expression (2.4*a*) for $\tilde{f}_L(x)$ is

$$\tilde{f}_L(x) = 1 + \alpha^{-1}\sum_{\ell=1}^{\infty}\alpha^{\ell d_2}x^{\ell d_2}\sum_{k=1}^{\ell}\frac{1}{k!}f_R^{(k)}(1)\Bigg(\sum_{|i|=\ell}b_{i_1}b_{i_2}\ldots b_{i_k}\Bigg). \tag{2.17}$$

We have

$$\tilde{a}_\ell = \alpha^{\ell d_2 - 1}\sum_{k=1}^{\ell}\frac{1}{k!}f_R^{(k)}(1)\Bigg(\sum_{|i|=\ell}b_{i_1}b_{i_2}\ldots b_{i_k}\Bigg). \tag{2.18}$$

In particular we have

$$\tilde{a}_1 = \alpha^{d_2-1}f_R'(1)b_1. \tag{2.19}$$

For there to be a fixed point of $T$ we clearly need $d_1 = d_2$. With $d_1 = d_2$, using equations (2.14) and (2.19), the conditions $\tilde{a}_1 = a_1$ and $\tilde{b}_1 = b_1$ imply that $a_1/b_1 = b_1/a_1$ and so $a_1 = b_1$ since they are of the same sign. By induction it follows that $a_\ell = b_\ell$ for all $\ell$. This is a consequence of equations (2.16) and (2.18).

We conclude that any fixed points of $T$ must be symmetric, i.e. have $f_L = f_R$.

We now consider period-two points of $T$. A second application of $T$ gives a new pair $(\tilde{\tilde{f}}_L, \tilde{\tilde{f}}_R)$ given by

$$\tilde{\tilde{f}}_L(x) = \tilde{\alpha}^{-1}\tilde{f}_R\tilde{f}_R(\tilde{\alpha}x) \tag{2.20a}$$

$$\tilde{\tilde{f}}_R(x) = \tilde{\alpha}^{-1}\tilde{f}_R\tilde{f}_L(\tilde{\alpha}x) \tag{2.20b}$$

where

$$\tilde{\alpha} = \tilde{f}_R(1). \tag{2.21}$$

We write these in the form

$$\tilde{\tilde{f}}_L(x) = 1 + \sum_{j=1}^{\infty}\tilde{\tilde{a}}_j x^{jd_1} \tag{2.22a}$$

$$\tilde{\tilde{f}}_R(x) = 1 + \sum_{j=1}^{\infty}\tilde{\tilde{b}}_j x^{jd_2} \tag{2.22b}$$

and we have

$$\tilde{\tilde{a}}_j = \tilde{\alpha}^{jd_1-1}\sum_{k=1}^{j}\frac{1}{k!}\tilde{f}_R^{(k)}(1)\Bigg(\sum_{|i|=j}\tilde{b}_{i_1}\tilde{b}_{i_2}\ldots\tilde{b}_{i_k}\Bigg) \tag{2.23a}$$

$$\tilde{\tilde{b}}_j(x) = \tilde{\alpha}^{jd_2-1}\sum_{k=1}^{j}\frac{1}{k!}\tilde{f}_R^{(k)}(1)\Bigg(\sum_{|i|=j}\tilde{a}_{i_1}\tilde{a}_{i_2}\ldots\tilde{a}_{i_k}\Bigg) \tag{2.23b}$$

where

$$\tilde{\alpha} = \tilde{f}_R(1) = 1 + \sum_{i=1}^{\infty} \tilde{b}_i \tag{2.24}$$

and

$$\frac{1}{k!} \tilde{f}_R^{(k)}(1) = \sum_{i=\left[\frac{k+d_1-1}{d_1}\right]}^{\infty} \binom{id_1}{k} \tilde{b}_i. \tag{2.25}$$

In order to study the period-two problem for $T$ we impose the conditions $\tilde{\tilde{b}}_j = b_j$ and $\tilde{\tilde{a}}_j = a_j$. The first term ($j = 1$) gives

$$\tilde{\tilde{a}}_1 = \tilde{\alpha}^{d_1-1} \tilde{f}_R'(1) \tilde{b}_1 \tag{2.26a}$$

$$\tilde{\tilde{b}}_1 = \tilde{\alpha}^{d_2-1} \tilde{f}_R'(1) \tilde{a}_1 \tag{2.26b}$$

where we have, by (2.14) and (2.19),

$$\tilde{a}_1 = \alpha^{d_2-1} f_R'(1) b_1 \tag{2.27a}$$

$$\tilde{b}_1 = \alpha^{d_1-1} f_R'(1) a_1. \tag{2.27b}$$

Thus the equations

$$\tilde{\tilde{a}}_1 = \tilde{\alpha}^{d_1-1} \tilde{f}_R'(1) \tilde{b}_1 = a_1 \tag{2.28a}$$

$$\tilde{\tilde{b}}_1 = \tilde{\alpha}^{d_2-1} \tilde{f}_R'(1) \tilde{a}_1 = b_1 \tag{2.28b}$$

become

$$\tilde{\alpha}^{d_1-1} \tilde{f}_R'(1) \alpha^{d_1-1} f_R'(1) a_1 = a_1 \tag{2.29a}$$

$$\tilde{\alpha}^{d_2-1} \tilde{f}_R'(1) \alpha^{d_2-1} f_R'(1) b_1 = b_1 \tag{2.29b}$$

which give

$$(\tilde{\alpha}\alpha)^{d_1-1} \tilde{f}_R'(1) f_R'(1) = 1 \tag{2.30a}$$

$$(\tilde{\alpha}\alpha)^{d_2-1} \tilde{f}_R'(1) f_R'(1) = 1. \tag{2.30b}$$

When $d_1 \neq d_2$ we deduce from these last two equations that

$$\tilde{\alpha}\alpha = 1. \tag{2.31}$$

We now consider briefly the question of whether period-two points can exist for $d_1 \neq d_2$. We shall return to the case $d_1 = d_2$ in the next section.

The period-two equations can be written as

$$f_L(x) = (\alpha\tilde{\alpha})^{-1} f_R f_L f_R f_L(\alpha\tilde{\alpha}x) \tag{2.32}$$

$$f_R(x) = (\alpha\tilde{\alpha})^{-1} f_R f_L f_R f_R(\alpha\tilde{\alpha}x) \tag{2.33}$$

and in particular, since $d_1 \neq d_2$, we have by (2.31)

$$f_R(x) = f_R f_L f_R f_R(x) \tag{2.34}$$

which can only be satisfied if $f_R f_L f_R$ is the identity function for $x$ close to 1. Now for unimodal maps 0 is the only point that maps to 1, and so $f_L f_R(1) = 0$. Differentiating at 1 gives $(f_R f_L f_R)'(1) = 0$ which contradicts $f_R f_L f_R$ being the identity function. Thus no analytic period-two points can exist.

This leaves open the question of what determines the metric bifurcation structure of families of unimodal maps with $d_1 \neq d_2$. One can draw bifurcation diagrams for our maps (2.1) when $d_1 \neq d_2$, and topologically all is as in the standard Feigenbaum case. However, as indicated by the numerical results of Jensen and Ma [11], there does not seem to be geometric scaling of Feigenbaum type. The fact that there are no fixed or period-two points of $T$ supports these findings.

## 3. An invariant modulus for the $d_1 = d_2$ case

Equations (2.14) and (2.19) show us that

$$f_R'(1) = \frac{\tilde{a}_1}{b_1\alpha^{d_2-1}} = \frac{\tilde{b}_1}{a_1\alpha^{d_1-1}} \tag{3.1}$$

and hence

$$\alpha^{d_1}a_1\tilde{a}_1 = \alpha^{d_2}b_1\tilde{b}_1. \tag{3.2}$$

Thus, when $d_1 = d_2 = d$ we deduce that

$$a_1\tilde{a}_1 = b_1\tilde{b}_1 \tag{3.3}$$

or equivalently

$$\frac{\tilde{a}_1}{\tilde{b}_1} = \frac{b_1}{a_1}. \tag{3.4}$$

We thus have an 'invariant modulus'

$$\mu = \frac{a_1}{b_1} \tag{3.5}$$

that inverts under one renormalization and then reverts to its original value on a second renormalization. This is similar to the invariant modulus introduced in [14] (see also [12]).

If we now consider the (second iterate) fixed point equation we find that the next terms satisfy

$$\frac{a_2}{b_2} = \left(\frac{a_1}{b_1}\right)^2 = \mu^2 \tag{3.6}$$

and

$$\tilde{f}_R''(1) = \frac{2\alpha}{a_1^2(\tilde{\alpha}\alpha)^{2d-1}f_R'(1)^2}\left(a_2(1-(\tilde{\alpha}\alpha)^d) - \frac{a_1^2(\tilde{\alpha}\alpha)^d f_R''(1)}{2f_R'(1)}\right). \tag{3.7}$$

Much more is true however. Using (2.16), (2.18) and (2.23), a straightforward induction argument shows that

$$\frac{a_\ell}{b_\ell} = \mu^\ell \tag{3.8}$$

for all $\ell$. We thus have

$$f_L(x) = 1 + \sum_{i=1}^{\infty} b_i\mu^i x^{id} = 1 + \sum_{i=1}^{\infty} b_i(\mu^{1/d}x)^{id} = f_R(\mu^{1/d}x) \tag{3.9}$$

$$f_R(x) = 1 + \sum_{i=1}^{\infty} b_i x^{id}. \tag{3.10}$$

We see that the fixed point pair can be expressed solely in terms of the single function $f_R$ and the invariant modulus $\mu$. Note that when $\mu = 1$ we have $f_L = f_R$ as expected.

It is clear from the preceding arguments that there is a simple symmetry between the two values of the modulus $\mu$ and $1/\mu$ as far as the scaling constants $\alpha$ and $\tilde{\alpha}$ are concerned—they are interchanged. Their product is consequently the same for parameters $\mu$ and $1/\mu$.

In the standard degree $d$ Feigenbaum situation (1.2), with corresponding fixed point equation

$$f(x) = \alpha^{-1} ff(\alpha x) \tag{3.11}$$

where $\alpha = f(1)$, on proposing the series solution

$$f(x) = 1 + \sum_{i=1}^{\infty} a_i x^{id} \tag{3.12}$$

we deduce that

$$f'(1) = \alpha^{1-d} \tag{3.13}$$

and

$$f''(1) = \frac{2a_2(1-\alpha^d)}{a_1^2 \alpha^{2d-1}}. \tag{3.14}$$

Our expressions above ((3.1), (3.7)) simplify to these when $\mu = 1$.

As a further indication of the existence of period-two points of $T$ we consider the linearization of the operator $T$ at the Feigenbaum fixed point.

Let $(f_L, f_R)$ be the degree $d$ fixed point of $T$, and let $L$ denote the derivative $\mathrm{d}T(f_L, f_R)$ at the fixed point. Let $C$ denote the linear map $C(g_L, g_R) = g_L - g_R$ and $(X_L, X_R)$ be an eigenvector of $L$ satisfying $C(X_L, X_R)(0) = 0$, but with $C(X_L, X_R)^{(d)} \neq 0$. Then the associated eigenvalue $\rho$ is $-1$.

To see this we note that $C$ is a linear map, so, taking the derivative at the fixed point $(f_L, f_R)$, we obtain

$$CL(X_L, X_R) = \rho C(X_L, X_R). \tag{3.15}$$

However, by standard results on functional differentiation, we have

$$\begin{aligned} CL(X_L, X_R)(x) = (&-\alpha^{-2} f_R(f_R(\alpha x)) + \alpha^{-1} f_R'(f_R(\alpha x)) f_R'(\alpha x)x)X_R(1) \\ &+\alpha^{-1} X_R(f_R(\alpha x)) + \alpha^{-1} f_R'(f_R(\alpha x)) X_R(\alpha x) \\ &-(-\alpha^{-2} f_R(f_L(\alpha x)) + \alpha^{-1} f_R'(f_L(\alpha x)) f_L'(\alpha x)x)X_R(1) \\ &-\alpha^{-1} X_R(f_L(\alpha x)) - \alpha^{-1} f_R'(f_L(\alpha x)) X_L(\alpha x) \end{aligned} \tag{3.16}$$

$$\begin{aligned} = {} &\alpha^{-1} X_R(1) C(f_R, f_L)(x) \\ &+\alpha^{-1} X_R(1)(f_R'(f_R(\alpha x)) f_R'(\alpha x)x - f_R'(f_L(\alpha x)) f_L'(\alpha x)x) \\ &+\alpha^{-1}(X_R(f_R(\alpha x)) - X_R(f_L(\alpha x))) \\ &+\alpha^{-1}(f_R'(f_R(\alpha x)) X_R(\alpha x) - f_R'(f_L(\alpha x)) X_L(\alpha x)) \end{aligned} \tag{3.17}$$

where we have used the fixed point equations. Differentiating $d$ times with respect to $x$ and evaluating at $x = 0$, we have, from the fact that $f_R^{(p)}(0)$ and $f_L^{(p)}(0)$ are zero for $p = 1, 2, \ldots, d-1$, and $C(X_L, X_R)(0) = 0$,

$$\begin{aligned} (CL(X_L, X_R))^{(d)}(0) = {} &\alpha^{-1} X_R(1) C(f_R, f_L)^{(d)}(0) + d\alpha^{d-2} X_R(1) f_R'(1) C(f_R, f_L)^{(d)}(0) \\ &+\alpha^{d-1} X_R'(1) C(f_R, f_L)^{(d)}(0) + \alpha^{d-1} f_R'(1) C(X_R, X_L)^{(d)}(0) \\ &+\alpha^{d-1} f_R''(1)(f_R^{(d)}(0) X_R(0) - f_L^{(d)}(0) X_L(0)) \end{aligned} \tag{3.18}$$

$$= -\alpha^{d-1} f_R'(1) C(X_L, X_R)^{(d)}(0) \tag{3.19}$$

where we have used the fact that $C(f_L, f_R)^{(d)}(0) = 0$ at the fixed point $(f_L, f_R)$. Now, as seen above (3.13), $f_R'(1) = \alpha^{1-d}$, so we have

$$\rho C(X_L, X_R)^{(d)}(0) = -C(X_L, X_R)^{(d)}(0) \tag{3.20}$$

giving $\rho = -1$ as required.

This eigenvalue $-1$ suggests that there is a line of period-two points of $T$ through the fixed point $(f_L, f_R)$ parametrized by $\mu$.

## 4. Numerical results

We now provide supporting numerical evidence for the existence and universality of the scaling. Similar calculations appear in [1] and [15]. However, there is an important difference between the scaling constants one calculates directly and those in our renormalization analysis.

Associated with the fixed point of Feigenbaum's renormalization operator (1.2) are two important scaling constants. The first of these, $\alpha$, is merely $f(1)$. The second, $\delta$, is the size of the largest eigenvalue of the linearization of the operator at the fixed point. In the case of quadratic critical points we have $\alpha = -0.399\,535\,28\ldots$ and $\delta = 4.669\,2016\ldots$. (For very precise calculations of these constants see [2].)

The renormalization picture shows us that $\delta$ is the (asymptotic) rate of convergence of parameter values at successive period-doubled superstable periodic orbits. It also tells us that $\alpha$ is the limit of the ratio of successive distances between the critical point and its partner point halfway round these orbits.

In light of the previous section, we take our map (1.1) and set $\lambda_1 = \mu\lambda_2$ and $\lambda_2 = \lambda$ fixing the ratio $\mu = \lambda_1/\lambda_2$ and the degree $d$. We then vary the single parameter $\lambda$ and observe period doubling. Our map is now

$$f_\lambda(x) = \begin{cases} 1 - \mu\lambda|x|^d & x \leqslant 0 \\ 1 - \lambda|x|^d & x \geqslant 0. \end{cases} \tag{4.1}$$

We define $\lambda_{(n)}$ to be the parameter value where there is a superstable periodic orbit (i.e. an orbit containing the point 0) of period $2^n$, and calculate the ratios

$$\delta_{(n)} = \frac{\lambda_{(n-1)} - \lambda_{(n-2)}}{\lambda_{(n)} - \lambda_{(n-1)}} \tag{4.2}$$

and

$$\alpha_{(n)} = \frac{f_{\lambda_{(n)}}^{2^{n-1}}(0)}{f_{\lambda_{(n-1)}}^{2^{n-2}}(0)}. \tag{4.3}$$

As in [1], we find that the limiting behaviour is period two, and we consequently define

$$\alpha^+ = \lim_{n\to\infty} \alpha_{(2n)} \qquad \text{and} \qquad \alpha^- = \lim_{n\to\infty} \alpha_{(2n+1)}. \tag{4.4}$$

In table 1 we tabulate these constants for several values of $\mu$ in the case of degree 2. The numbers quoted are accurate to five digits.

Note that $\alpha_\mu^+ = \alpha_{1/\mu}^-$, so that the results at parameter $\mu$ will be identical to those at $1/\mu$ with the roles of $\alpha^+$ and $\alpha^-$ reversed The same remarks hold for the corresponding limits $\delta^+$ and $\delta^-$.

Figure 1 shows $\sqrt{\alpha^+\alpha^-}$ and $\sqrt{\delta^+\delta^-}$ against $\log\mu$, the logarithmic scale being chosen to exhibit the symmetry with respect to $\mu$.

**Table 1.** Results for $d = 2$.

| $\mu$ | $-\alpha^+$ | $-\alpha^-$ | $\sqrt{\alpha^+\alpha^-}$ | $-\alpha$ | $-\tilde{\alpha}$ | $\delta^+$ | $\delta^-$ | $\sqrt{\delta^+\delta^-}$ |
|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.715 759 | 0.218 862 | 0.395 794 | 0.581 439 | 0.269 423 | 8.342 155 | 2.723 239 | 4.766 307 |
| 0.6 | 0.614 996 | 0.256 911 | 0.397 491 | 0.527 784 | 0.299 363 | 7.136 768 | 3.124 109 | 4.721 868 |
| 0.7 | 0.540 480 | 0.293 868 | 0.398 535 | 0.485 794 | 0.326 949 | 6.265 819 | 3.517 770 | 4.694 860 |
| 0.8 | 0.482 941 | 0.329 885 | 0.399 143 | 0.451 780 | 0.352 638 | 5.605 785 | 3.905 832 | 4.679 237 |
| 0.9 | 0.437 055 | 0.365 077 | 0.399 448 | 0.423 509 | 0.376 753 | 5.087 494 | 4.289 389 | 4.671 428 |
| 1.0 | 0.399 535 | 0.399 535 | 0.399 535 | 0.399 535 | 0.399 535 | 4.669 206 | 4.669 200 | 4.669 203 |
| 1.1 | 0.368 238 | 0.433 337 | 0.399 464 | 0.378 876 | 0.421 170 | 4.324 062 | 5.045 843 | 4.671 032 |
| 1.2 | 0.341 702 | 0.466 544 | 0.399 273 | 0.360 838 | 0.441 802 | 4.034 146 | 5.419 750 | 4.675 902 |
| 1.3 | 0.318 896 | 0.499 209 | 0.398 993 | 0.344 915 | 0.461 550 | 3.786 959 | 5.791 255 | 4.683 081 |
| 1.4 | 0.299 068 | 0.531 376 | 0.398 645 | 0.330 727 | 0.480 510 | 3.573 526 | 6.160 631 | 4.692 033 |
| 1.5 | 0.281 660 | 0.563 083 | 0.398 244 | 0.317 984 | 0.498 761 | 3.387 243 | 6.528 093 | 4.702 365 |

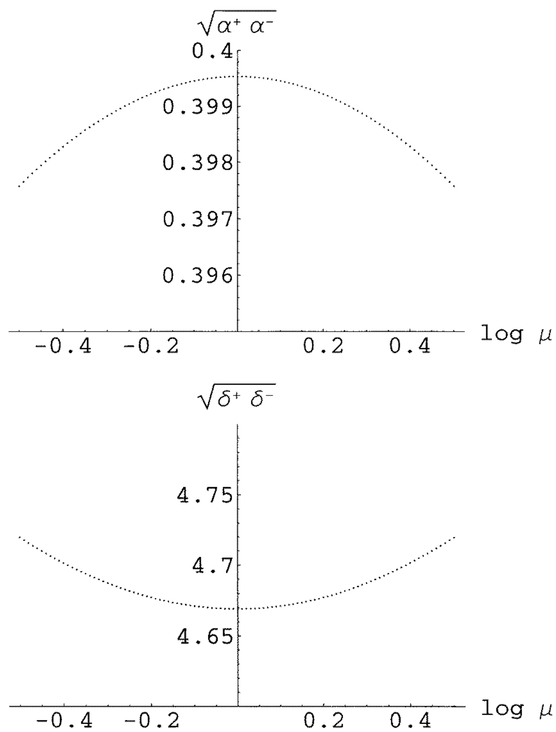


**Figure 1.** Results for $d = 2$. (*a*) $\sqrt{\alpha^+\alpha^-}$, (*b*) $\sqrt{\delta^+\delta^-}$.

As a check on the universality of these numbers we have also considered a different 'asymmetric quadratic' map. The map used was a simple variation on the logistic map of the interval:

$$f(x) = \begin{cases} p_1x + p_2x^2 + p_3x^3 & 0 \leqslant x \leqslant \frac{1}{2} \\ \lambda x - \lambda x^2 & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \tag{4.5}$$

with $p_1 = (3-\mu)\lambda/2$, $p_2 = (2\mu - 3)\lambda$, and $p_3 = 2(1-\mu)\lambda$. The modulus $\mu$ has identical meaning as above. The results obtained were identical.

It is an important interesting obversation that although $\alpha^+$ and $\alpha^-$ are universal they are *not* the (universal) quantities $\alpha$ and $\tilde{\alpha}$ encountered in previous sections.

In fact $\alpha^+ = \alpha A$ and $\alpha^- = \tilde{\alpha}/A$ where

$$A = \lim_{k\to\infty} \prod_{j=2}^{2k-1} \left(\frac{\alpha_j}{\tilde{\alpha}_j}\right)^{(-1)^j} \tag{4.6}$$

and where $\alpha_j$ and $\tilde{\alpha}_j$ are respectively the values of the scaling parameters $\alpha$ and $\tilde{\alpha}$ at the intersections of the $2^j$-superstable manifolds with the unstable manifolds of the period-two points $(f_L, f_R)$ and $(\tilde{f}_L, \tilde{f}_R)$. To calculate $\alpha$ and $\tilde{\alpha}$ one must calculate the ratios $f_{\lambda_{(\infty)}}^{2^n}(0)/f_{\lambda_{(\infty)}}^{2^{(n-1)}}(0)$ instead of the ratios in (4.3), where $\lambda_{(\infty)}$ is the parameter value at the accumulation of period-doubling. Table 1 also tabulates the values of $\alpha$ and $\tilde{\alpha}$ for comparison.

This result differs from the standard Feigenbaum case ($\mu = 1$) where $\alpha^+ = \alpha^- = \alpha = \tilde{\alpha}$. In general all we can deduce is that the product $\alpha^+\alpha^-$ is equal to $\alpha\tilde{\alpha}$. This observation indicates that care must be taken in interpreting the results of numerical calculations even when the results are universal.

## 5. Conclusion

In this paper we have shown that there is good numerical and theoretical evidence that suggests that the scaling behaviour for unimodal maps with asymmetric critical points of the same degree is governed by period-two points of the renormalization operator $T$.

We therefore conjecture that for each $0 < d < \infty$ and each $0 < \mu < \infty$ there exists a period-two pair $(f_L, f_R)$ of $T$ such that $f_L$ and $f_R$ are analytic functions of $|x|^d$: $f_L(x) = F_L(|x|^d)$, $f_R(x) = F_R(|x|^d)$, with $F_L(x) = F_R(\mu x)$, and $F_L'(0)/F_R'(0) = \mu$.

It is likely that the Herglotz function approach of Epstein [8] can be adapted to prove that such a pair exists.

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